Modified couple stress theory in orthogonal curvilinear coordinates

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Abstract

The formulations for the modified couple stress theory (MCST) are consistently derived in general orthogonal curvilinear coordinate systems. In particular, the expressions for the rotation vector, higher-order strain and stress tensors, i.e. the rotation gradient tensor and the deviatoric part of the symmetric couple stress tensor, and the classical strain and stress tensors are derived for an arbitrary orthogonal curvilinear coordinate system. Additionally, using the theory of surfaces, the formulations for the MCST are derived for general doubly curved coordinates, which are more convenient to use for shells of arbitrary curvature. The expressions for special cases, i.e. cylindrical and spherical shells, are obtained. The MCST expressions derived in this study are comprehensive and general, and can be used for consistent utilisation of the MCST in any orthogonal curvilinear coordinate system.

keywords: *modified couple stress theory*; *orthogonal curvilinear coordinate; doubly curved shell; geometric nonlinearity*

1. Introduction

In the last few decades, many higher-order continuum theories have been developed, which are mostly extensions or modifications of strain gradient theories developed by Toupin [1] and Mindlin [2-4]. The motivation behind development of such higher-order continuum theories is to account for size-effects in micro- and nano-scales, which the classical continuum theories are incapable of. These higher-order theories possess one or more intrinsic length-scale parameter which enables them to account for micro/nano-scale size-effects. Among the more recently introduced higher-order continuum theories, the modified couple stress theory (MCST) [5] is one of the most popular since it introduces only one additional characteristic length-scale parameter, compared to the classical continuum theory, to account for size-dependency. This theory is in fact a modification of the classical couple stress theories [1, 2]. The MCST has been widely used by different researchers for theoretical prediction of the size-dependent behaviour of microstructures. Furthermore, the MCST has been validated by comparing theoretical predictions to experimental ones [6].

The formulations of the MCST proposed by Yang et al. [5] is within a Cartesian coordinate system. Hence, it is usually a straightforward procedure to derive the MCST formulations for a certain problem in a rectangular Cartesian coordinate system. However, this procedure becomes nontrivial and complicated when dealing with curvilinear coordinate systems. Hence, consistent derivation of the MCST in general orthogonal curvilinear coordinate systems is of significant importance for laying the groundwork for any study involving curved elements. In this study, for the first time, the complete formulations for the modified version of the couple stress theory are derived in a general orthogonal curvilinear coordinates system.

Additionally, the expressions are derived for a general doubly curved coordinate, which are more convenient to be used for modelling of shells of arbitrary curvature and radius. Furthermore, the MSCT formulations are given for example cases, i.e. cylindrical and spherical shells, and can be readily used in relevant problems. It should be noted that the formulations developed in this study account for geometric nonlinearities arising from large deformations.

2. Modified couple stress theory in Cartesian coordinate system

According to the modified couples stress theory, apart from classical strains and stresses, i.e. ε_{ij} and σ_{ij}, respectively, higher-order strains and stresses are present as well in the material element. The higher-order strain tensor, i.e. the symmetric rotation gradient, is denoted by χ_{ij}, and the higher-order stress tensor, i.e. the deviatoric part of the symmetric couple stress, is represented by m_{ij}. In a rectangular Cartesian coordinate system (x_1,x_2,x_3), the symmetric rotation gradient tensor components can be expressed as

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial\omega_i}{\partial x_j} + \frac{\partial\omega_j}{\partial x_i}\right), \qquad i,j \in \{1,2,3\}, \tag{1}$$

in which ω_i denotes the rotation vector components given by

$$\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix} = \frac{1}{2}\nabla\times\mathbf{u} = \frac{1}{2}\begin{Bmatrix}\left(\partial u_3/\partial x_2 - \partial u_2/\partial x_3\right)\\ \left(\partial u_1/\partial x_3 - \partial u_3/\partial x_1\right)\\ \left(\partial u_2/\partial x_1 - \partial u_1/\partial x_2\right)\end{Bmatrix}, \tag{2}$$

where **u** is the displacement vector.

The nonlinear Green strain tensor components can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \right), \qquad i, j \in \{1,2,3\}. \tag{3}$$

In the framework of the MCST and for a linear elastic isotropic material, the relationship between stresses and strains are given by ($i, j \in \{1,2,3\}$)

$$\begin{aligned} \sigma_{ij} &= \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \\ m_{ij} &= 2\mu l^2 \chi_{ij}, \end{aligned} \tag{4}$$

in which l is the intrinsic length-scale parameter while λ and μ are Lamé constants given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{5}$$

In the following sections, the expressions given in Eqs. (1)-(3) are derived in general orthogonal curvilinear coordinate systems.

3. Modified couple stress theory in a general orthogonal curvilinear coordinate system

The general formulations for the MCST in an orthogonal curvilinear coordinates are derived, following the procedure introduced by Eringen [7] for transformations from Cartesian to curvilinear coordinate systems. To this end, a general orthogonal curvilinear coordinate system of coordinate lines ψ_1, ψ_2, ψ_3, is considered which is related to the Cartesian coordinates x_1, x_2, x_3 through

$$x_1 = f(\psi_1, \psi_2, \psi_3), \qquad x_2 = g(\psi_1, \psi_2, \psi_3), \qquad x_3 = h(\psi_1, \psi_2, \psi_3). \tag{6}$$

The square of the arc length can be expressed as [7]

$$\mathrm{d}s^2 = g_{11} \mathrm{d}\psi_1^{\,2} + g_{22} \mathrm{d}\psi_2^{\,2} + g_{33} \mathrm{d}\psi_3^{\,2}, \tag{7}$$

in which g_{11}, g_{22}, and g_{33} are the components of the Euclidean metric tensor given by

$$g_{11}=\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial\psi_1}\right)^2=\left[\left(\frac{\partial x_1}{\partial\psi_1}\right)^2+\left(\frac{\partial x_2}{\partial\psi_1}\right)^2+\left(\frac{\partial x_3}{\partial\psi_1}\right)^2\right],$$
$$g_{22}=\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial\psi_2}\right)^2=\left[\left(\frac{\partial x_1}{\partial\psi_2}\right)^2+\left(\frac{\partial x_2}{\partial\psi_2}\right)^2+\left(\frac{\partial x_3}{\partial\psi_2}\right)^2\right], \tag{8}$$
$$g_{33}=\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial\psi_3}\right)^2=\left[\left(\frac{\partial x_1}{\partial\psi_3}\right)^2+\left(\frac{\partial x_2}{\partial\psi_3}\right)^2+\left(\frac{\partial x_3}{\partial\psi_3}\right)^2\right].$$

In the present study, the notations $H_1=\sqrt{g_{11}}$, $H_2=\sqrt{g_{22}}$, and $H_3=\sqrt{g_{33}}$ are used for the sake of clarity.

The rotation vector (**ω**) components in an orthogonal curvilinear coordinate system can be obtained as

$$\omega_k=\frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{e_{kji}^{(p)}}{H_iH_j}\left[\frac{\partial}{\partial\psi_j}\left(H_iu_i\right)-\frac{\partial}{\partial\psi_i}\left(H_ju_j\right)\right], \tag{9}$$

where $e^{(p)}$ is the permutation symbol and u_i and u_j denote the displacement vector (**u**) components. Simplifying Eq. (9), the components of the rotation vector **ω** in an arbitrary orthogonal curvilinear coordinate system are derived as

$$\omega_1=\frac{1}{2}\left[\frac{1}{H_2}\frac{\partial u_3}{\partial\psi_2}-\frac{1}{H_3}\frac{\partial u_2}{\partial\psi_3}+\frac{1}{H_2H_3}\left(\frac{\partial H_3}{\partial\psi_2}u_3-\frac{\partial H_2}{\partial\psi_3}u_2\right)\right],$$
$$\omega_2=\frac{1}{2}\left[\frac{1}{H_3}\frac{\partial u_1}{\partial\psi_3}-\frac{1}{H_1}\frac{\partial u_3}{\partial\psi_1}+\frac{1}{H_1H_3}\left(\frac{\partial H_1}{\partial\psi_3}u_1-\frac{\partial H_3}{\partial\psi_1}u_3\right)\right], \tag{10}$$
$$\omega_3=\frac{1}{2}\left[\frac{1}{H_1}\frac{\partial u_2}{\partial\psi_1}-\frac{1}{H_2}\frac{\partial u_1}{\partial\psi_2}+\frac{1}{H_1H_2}\left(\frac{\partial H_2}{\partial\psi_1}u_2-\frac{\partial H_1}{\partial\psi_2}u_1\right)\right].$$

Following the rules defined by Eringen [7] for transformations from Cartesian coordinate system to an orthogonal curvilinear one, the symmetric rotation gradient tensor ($\boldsymbol{\chi}$) components in the orthogonal curvilinear coordinate system can be derived as

$$\chi_{ij}=\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{H_i}{H_j}\frac{\partial}{\partial\psi_j}\left(\frac{\omega_i}{H_i}\right)+\frac{H_j}{H_i}\frac{\partial}{\partial\psi_i}\left(\frac{\omega_j}{H_j}\right)+\frac{1}{H_iH_j}\sum_{k=1}^{3}\frac{\partial H_i}{\partial\psi_k}\frac{\omega_k}{H_k}\delta_{ij}\right], \tag{11}$$

which yields

$$\begin{aligned}
\chi_{11}&=\frac{1}{H_1}\frac{\partial\omega_1}{\partial\psi_1}+\frac{\omega_2}{H_1H_2}\frac{\partial H_1}{\partial\psi_2}+\frac{\omega_3}{H_1H_3}\frac{\partial H_1}{\partial\psi_3},\\
\chi_{22}&=\frac{1}{H_2}\frac{\partial\omega_2}{\partial\psi_2}+\frac{\omega_1}{H_1H_2}\frac{\partial H_2}{\partial\psi_1}+\frac{\omega_3}{H_2H_3}\frac{\partial H_2}{\partial\psi_3},\\
\chi_{33}&=\frac{1}{H_3}\frac{\partial\omega_3}{\partial\psi_3}+\frac{\omega_1}{H_3H_1}\frac{\partial H_3}{\partial\psi_1}+\frac{\omega_2}{H_3H_2}\frac{\partial H_3}{\partial\psi_2},\\
\chi_{12}&=\frac{1}{2}\left[\frac{1}{H_2}\frac{\partial\omega_1}{\partial\psi_2}+\frac{1}{H_1}\frac{\partial\omega_2}{\partial\psi_1}-\frac{1}{H_1H_2}\left(\frac{\partial H_1}{\partial\psi_2}\omega_1+\frac{\partial H_2}{\partial\psi_1}\omega_2\right)\right],\\
\chi_{13}&=\frac{1}{2}\left[\frac{1}{H_3}\frac{\partial\omega_1}{\partial\psi_3}+\frac{1}{H_1}\frac{\partial\omega_3}{\partial\psi_1}-\frac{1}{H_1H_3}\left(\frac{\partial H_1}{\partial\psi_3}\omega_1+\frac{\partial H_3}{\partial\psi_1}\omega_3\right)\right],\\
\chi_{23}&=\frac{1}{2}\left[\frac{1}{H_3}\frac{\partial\omega_2}{\partial\psi_3}+\frac{1}{H_2}\frac{\partial\omega_3}{\partial\psi_2}-\frac{1}{H_2H_3}\left(\frac{\partial H_2}{\partial\psi_3}\omega_2+\frac{\partial H_3}{\partial\psi_2}\omega_3\right)\right].
\end{aligned} \tag{12}$$

Substituting Eq. (10) into Eq. (12) gives the components of the rotation gradient tensor in terms of the displacement components as

$$\begin{aligned}
\chi_{11}=&\frac{1}{2}\frac{1}{H_1H_2H_3}\left[\frac{\partial^2}{\partial\psi_1\partial\psi_2}(H_3u_3)-\frac{\partial^2}{\partial\psi_1\partial\psi_3}(H_2u_2)\right]+\frac{1}{2}\frac{1}{H_1}\frac{\partial}{\partial\psi_1}\left(\frac{1}{H_2H_3}\right)\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\\
&+\frac{1}{2}\frac{1}{H_1^2H_2H_3}\left\{\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\frac{\partial H_1}{\partial\psi_2}+\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]\frac{\partial H_1}{\partial\psi_3}\right\},
\end{aligned}$$

$$\begin{aligned}
\chi_{22}=&\frac{1}{2}\frac{1}{H_1H_2H_3}\left[\frac{\partial^2}{\partial\psi_2\partial\psi_3}(H_1u_1)-\frac{\partial^2}{\partial\psi_1\partial\psi_2}(H_3u_3)\right]+\frac{1}{2}\frac{1}{H_2}\frac{\partial}{\partial\psi_2}\left(\frac{1}{H_1H_3}\right)\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\\
&+\frac{1}{2}\frac{1}{H_1H_2^2H_3}\left\{\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\frac{\partial H_2}{\partial\psi_1}+\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]\frac{\partial H_2}{\partial\psi_3}\right\},
\end{aligned}$$

$$
\chi_{33}=\frac{1}{2}\frac{1}{H_1H_2H_3}\left[\frac{\partial^2}{\partial\psi_1\partial\psi_3}(H_2u_2)-\frac{\partial^2}{\partial\psi_2\partial\psi_3}(H_1u_1)\right]+\frac{1}{2}\frac{1}{H_3}\frac{\partial}{\partial\psi_3}\left(\frac{1}{H_1H_2}\right)\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]
$$

$$
+\frac{1}{2}\frac{1}{H_1H_2H_3^2}\left\{\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\frac{\partial H_3}{\partial\psi_1}+\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\frac{\partial H_3}{\partial\psi_2}\right\},
$$

$$
\chi_{12}=\frac{1}{4}\frac{1}{H_2}\frac{\partial}{\partial\psi_2}\left(\frac{1}{H_2H_3}\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\right)+\frac{1}{4}\frac{1}{H_1}\frac{\partial}{\partial\psi_1}\left(\frac{1}{H_3H_1}\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\right)
$$

$$
-\frac{1}{4}\frac{1}{H_1H_2H_3}\left\{\frac{1}{H_2}\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\frac{\partial H_1}{\partial\psi_2}+\frac{1}{H_1}\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\frac{\partial H_2}{\partial\psi_1}\right\},
$$

$$
\chi_{13}=\frac{1}{4}\frac{1}{H_3}\frac{\partial}{\partial\psi_3}\left(\frac{1}{H_2H_3}\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\right)+\frac{1}{4}\frac{1}{H_1}\frac{\partial}{\partial\psi_1}\left(\frac{1}{H_1H_2}\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]\right)
$$

$$
-\frac{1}{4}\frac{1}{H_1H_2H_3}\left(\frac{1}{H_3}\left[\frac{\partial}{\partial\psi_2}(H_3u_3)-\frac{\partial}{\partial\psi_3}(H_2u_2)\right]\frac{\partial H_1}{\partial\psi_3}+\frac{1}{H_1}\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]\frac{\partial H_3}{\partial\psi_1}\right),
$$

$$
\chi_{23}=\frac{1}{4}\frac{1}{H_3}\frac{\partial}{\partial\psi_3}\left(\frac{1}{H_3H_1}\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\right)+\frac{1}{4}\frac{1}{H_2}\frac{\partial}{\partial\psi_2}\left(\frac{1}{H_1H_2}\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]\right)
$$

$$
-\frac{1}{4}\frac{1}{H_1H_2H_3}\left\{\frac{1}{H_3}\left[\frac{\partial}{\partial\psi_3}(H_1u_1)-\frac{\partial}{\partial\psi_1}(H_3u_3)\right]\frac{\partial H_2}{\partial\psi_3}+\frac{1}{H_2}\left[\frac{\partial}{\partial\psi_1}(H_2u_2)-\frac{\partial}{\partial\psi_2}(H_1u_1)\right]\frac{\partial H_3}{\partial\psi_2}\right\}.
$$

(13)

Following a similar procedure for the strain tensor gives its linear components (e_{ij}) in the orthogonal curvilinear coordinate system as

$$
\begin{aligned}
e_{11} &= \frac{1}{H_1}\frac{\partial u_1}{\partial \psi_1} + \frac{u_2}{H_1 H_2}\frac{\partial H_1}{\partial \psi_2} + \frac{u_3}{H_1 H_3}\frac{\partial H_1}{\partial \psi_3}, \\
e_{22} &= \frac{1}{H_2}\frac{\partial u_2}{\partial \psi_2} + \frac{u_1}{H_1 H_2}\frac{\partial H_2}{\partial \psi_1} + \frac{u_3}{H_2 H_3}\frac{\partial H_2}{\partial \psi_3}, \\
e_{33} &= \frac{1}{H_3}\frac{\partial u_3}{\partial \psi_3} + \frac{u_1}{H_1 H_3}\frac{\partial H_3}{\partial \psi_1} + \frac{u_2}{H_2 H_3}\frac{\partial H_3}{\partial \psi_2}, \\
e_{12} &= \frac{1}{2}\left[\frac{1}{H_2}\frac{\partial u_1}{\partial \psi_2} + \frac{1}{H_1}\frac{\partial u_2}{\partial \psi_1} - \frac{1}{H_1 H_2}\left(\frac{\partial H_1}{\partial \psi_2}u_1 + \frac{\partial H_2}{\partial \psi_1}u_2\right)\right], \\
e_{13} &= \frac{1}{2}\left[\frac{1}{H_3}\frac{\partial u_1}{\partial \psi_3} + \frac{1}{H_1}\frac{\partial u_3}{\partial \psi_1} - \frac{1}{H_1 H_3}\left(\frac{\partial H_1}{\partial \psi_3}u_1 + \frac{\partial H_3}{\partial \psi_1}u_3\right)\right], \\
e_{23} &= \frac{1}{2}\left[\frac{1}{H_3}\frac{\partial u_2}{\partial \psi_3} + \frac{1}{H_2}\frac{\partial u_3}{\partial \psi_2} - \frac{1}{H_2 H_3}\left(\frac{\partial H_2}{\partial \psi_3}u_2 + \frac{\partial H_3}{\partial \psi_2}u_3\right)\right].
\end{aligned}
\tag{14}
$$

The nonlinear Green strain tensor components can be expressed via the linear strain and the rotation vector components as

$$
\begin{aligned}
\varepsilon_{11} &= e_{11} + \frac{1}{2}\left[\left(e_{11}\right)^2 + \left(e_{12}+\omega_3\right)^2 + \left(e_{13}-\omega_2\right)^2\right], \\
\varepsilon_{22} &= e_{22} + \frac{1}{2}\left[\left(e_{22}\right)^2 + \left(e_{12}-\omega_3\right)^2 + \left(e_{23}+\omega_1\right)^2\right], \\
\varepsilon_{33} &= e_{33} + \frac{1}{2}\left[\left(e_{33}\right)^2 + \left(e_{13}+\omega_2\right)^2 + \left(e_{23}-\omega_1\right)^2\right], \\
\varepsilon_{12} &= e_{12} + \frac{1}{2}\left[e_{11}\left(e_{12}-\omega_3\right) + e_{22}\left(e_{12}+\omega_3\right) + \left(e_{13}-\omega_2\right)\left(e_{23}+\omega_1\right)\right], \\
\varepsilon_{13} &= e_{13} + \frac{1}{2}\left[e_{11}\left(e_{13}+\omega_2\right) + e_{33}\left(e_{13}-\omega_2\right) + \left(e_{12}+\omega_3\right)\left(e_{23}-\omega_1\right)\right], \\
\varepsilon_{23} &= e_{23} + \frac{1}{2}\left[e_{22}\left(e_{23}-\omega_1\right) + e_{33}\left(e_{23}+\omega_1\right) + \left(e_{12}-\omega_3\right)\left(e_{13}+\omega_2\right)\right],
\end{aligned}
\tag{15}
$$

which can be expanded and rewritten in terms of the displacement components as

$$
\begin{aligned}
\varepsilon_{11} = {} & \frac{1}{H_1}\frac{\partial u_1}{\partial \psi_1} + \frac{u_2}{H_1 H_2}\frac{\partial H_1}{\partial \psi_2} + \frac{u_3}{H_1 H_3}\frac{\partial H_1}{\partial \psi_3} + \frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_1}{\partial \psi_1} + \frac{u_2}{H_1 H_2}\frac{\partial H_1}{\partial \psi_2} + \frac{u_3}{H_1 H_3}\frac{\partial H_1}{\partial \psi_3}\right)^2 \\
& + \frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_2}{\partial \psi_1} - \frac{u_1}{H_1 H_2}\frac{\partial H_1}{\partial \psi_2}\right)^2 + \frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_3}{\partial \psi_1} - \frac{u_1}{H_1 H_3}\frac{\partial H_1}{\partial \psi_3}\right)^2,
\end{aligned}
$$

$$\varepsilon_{22}=\frac{1}{H_2}\frac{\partial u_2}{\partial \psi_2}+\frac{u_1}{H_1H_2}\frac{\partial H_2}{\partial \psi_1}+\frac{u_3}{H_2H_3}\frac{\partial H_2}{\partial \psi_3}+\frac{1}{2}\left(\frac{1}{H_2}\frac{\partial u_2}{\partial \psi_2}+\frac{u_1}{H_1H_2}\frac{\partial H_2}{\partial \psi_1}+\frac{u_3}{H_2H_3}\frac{\partial H_2}{\partial \psi_3}\right)^2$$
$$+\frac{1}{2}\left(\frac{1}{H_2}\frac{\partial u_1}{\partial \psi_2}-\frac{u_2}{H_1H_2}\frac{\partial H_2}{\partial \psi_1}\right)^2+\frac{1}{2}\left(\frac{1}{H_2}\frac{\partial u_3}{\partial \psi_2}-\frac{u_2}{H_2H_3}\frac{\partial H_2}{\partial \psi_3}\right)^2,$$

$$\varepsilon_{33}=\frac{1}{H_3}\frac{\partial u_3}{\partial \psi_3}+\frac{u_1}{H_1H_3}\frac{\partial H_3}{\partial \psi_1}+\frac{u_2}{H_2H_3}\frac{\partial H_3}{\partial \psi_2}+\frac{1}{2}\left(\frac{1}{H_3}\frac{\partial u_3}{\partial \psi_3}+\frac{u_1}{H_1H_3}\frac{\partial H_3}{\partial \psi_1}+\frac{u_2}{H_2H_3}\frac{\partial H_3}{\partial \psi_2}\right)^2 \tag{16}$$
$$+\frac{1}{2}\left(\frac{1}{H_3}\frac{\partial u_1}{\partial \psi_3}-\frac{u_3}{H_1H_3}\frac{\partial H_3}{\partial \psi_1}\right)^2+\frac{1}{2}\left(\frac{1}{H_3}\frac{\partial u_2}{\partial \psi_3}-\frac{u_3}{H_2H_3}\frac{\partial H_3}{\partial \psi_2}\right)^2,$$

$$\varepsilon_{12}=\frac{1}{2}\left[\frac{1}{H_2}\frac{\partial u_1}{\partial \psi_2}+\frac{1}{H_1}\frac{\partial u_2}{\partial \psi_1}-\frac{1}{H_1H_2}\left(\frac{\partial H_1}{\partial \psi_2}u_1+\frac{\partial H_2}{\partial \psi_1}u_2\right)\right]$$
$$+\frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_1}{\partial \psi_1}+\frac{u_2}{H_1H_2}\frac{\partial H_1}{\partial \psi_2}+\frac{u_3}{H_1H_3}\frac{\partial H_1}{\partial \psi_3}\right)\left(\frac{1}{H_2}\frac{\partial u_1}{\partial \psi_2}-\frac{u_2}{H_1H_2}\frac{\partial H_2}{\partial \psi_1}\right)$$
$$+\frac{1}{2}\left(\frac{1}{H_2}\frac{\partial u_2}{\partial \psi_2}+\frac{u_1}{H_1H_2}\frac{\partial H_2}{\partial \psi_1}+\frac{u_3}{H_2H_3}\frac{\partial H_2}{\partial \psi_3}\right)\left(\frac{1}{H_1}\frac{\partial u_2}{\partial \psi_1}-\frac{u_1}{H_1H_2}\frac{\partial H_1}{\partial \psi_2}\right)$$
$$+\frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_3}{\partial \psi_1}-\frac{u_1}{H_1H_3}\frac{\partial H_1}{\partial \psi_3}\right)\left(\frac{1}{H_2}\frac{\partial u_3}{\partial \psi_2}-\frac{u_2}{H_2H_3}\frac{\partial H_2}{\partial \psi_3}\right),$$

$$\varepsilon_{13}=\frac{1}{2}\left[\frac{1}{H_3}\frac{\partial u_1}{\partial \psi_3}+\frac{1}{H_1}\frac{\partial u_3}{\partial \psi_1}-\frac{1}{H_1H_3}\left(\frac{\partial H_1}{\partial \psi_3}u_1+\frac{\partial H_3}{\partial \psi_1}u_3\right)\right]$$
$$+\frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_1}{\partial \psi_1}+\frac{u_2}{H_1H_2}\frac{\partial H_1}{\partial \psi_2}+\frac{u_3}{H_1H_3}\frac{\partial H_1}{\partial \psi_3}\right)\left(\frac{1}{H_3}\frac{\partial u_1}{\partial \psi_3}-\frac{u_3}{H_1H_3}\frac{\partial H_3}{\partial \psi_1}\right)$$
$$+\frac{1}{2}\left(\frac{1}{H_3}\frac{\partial u_3}{\partial \psi_3}+\frac{u_1}{H_1H_3}\frac{\partial H_3}{\partial \psi_1}+\frac{u_2}{H_2H_3}\frac{\partial H_3}{\partial \psi_2}\right)\left(\frac{1}{H_1}\frac{\partial u_3}{\partial \psi_1}-\frac{u_1}{H_1H_3}\frac{\partial H_1}{\partial \psi_3}\right)$$
$$+\frac{1}{2}\left(\frac{1}{H_1}\frac{\partial u_2}{\partial \psi_1}-\frac{u_1}{H_1H_2}\frac{\partial H_1}{\partial \psi_2}\right)\left(\frac{1}{H_3}\frac{\partial u_2}{\partial \psi_3}-\frac{u_3}{H_2H_3}\frac{\partial H_3}{\partial \psi_2}\right),$$

$$
\begin{aligned}
\varepsilon_{23} = \frac{1}{2}\left[\frac{1}{H_3}\frac{\partial u_2}{\partial \psi_3} + \frac{1}{H_2}\frac{\partial u_3}{\partial \psi_2} - \frac{1}{H_2 H_3}\left(\frac{\partial H_2}{\partial \psi_3}u_2 + \frac{\partial H_3}{\partial \psi_2}u_3\right)\right] \\
+\frac{1}{2}\left(\frac{1}{H_2}\frac{\partial u_2}{\partial \psi_2} + \frac{u_1}{H_1 H_2}\frac{\partial H_2}{\partial \psi_1} + \frac{u_3}{H_2 H_3}\frac{\partial H_2}{\partial \psi_3}\right)\left(\frac{1}{H_3}\frac{\partial u_2}{\partial \psi_3} - \frac{u_3}{H_2 H_3}\frac{\partial H_3}{\partial \psi_2}\right) \\
+\frac{1}{2}\left(\frac{1}{H_3}\frac{\partial u_3}{\partial \psi_3} + \frac{u_1}{H_1 H_3}\frac{\partial H_3}{\partial \psi_1} + \frac{u_2}{H_2 H_3}\frac{\partial H_3}{\partial \psi_2}\right)\left(\frac{1}{H_2}\frac{\partial u_3}{\partial \psi_2} - \frac{u_2}{H_2 H_3}\frac{\partial H_2}{\partial \psi_3}\right) \\
+\frac{1}{2}\left(\frac{1}{H_2}\frac{\partial u_1}{\partial \psi_2} - \frac{u_2}{H_1 H_2}\frac{\partial H_2}{\partial \psi_1}\right)\left(\frac{1}{H_3}\frac{\partial u_1}{\partial \psi_3} - \frac{u_3}{H_1 H_3}\frac{\partial H_3}{\partial \psi_1}\right).
\end{aligned}
$$

Inserting Eqs. (13) and (16), into Eq. (4), yields the classical and higher-order stress components.

4. Modified couple stress theory in orthogonal doubly curved coordinate systems

In this section, the MCST formulations are derived for an orthogonal doubly curved coordinate system. Although the orthogonal doubly curved coordinate system is more specific than the general coordinate system considered in the previous section, it is of more practical use, especially for modelling of shells. Nevertheless, the doubly curved coordinate system is still general enough to be applicable for various problems involving double curvatures.

Consider a doubly curved orthogonal curvilinear coordinate system (α_1, α_2, z), as shown in Fig. 1. It should be noted that in such a coordinate system, the coordinate line z is always perpendicular to coordinate lines α_1 and α_2 and hence to the surface. For this coordinate system, the Euclidian metric tensor components become

$$
H_1 = A_1\left(1+\frac{z}{R_1}\right), \qquad H_2 = A_2\left(1+\frac{z}{R_2}\right), \qquad H_3 = 1, \tag{17}
$$

in which A_1, A_2, R_1, and R_2 are functions of the curvilinear coordinates α_1, α_2. These functions must satisfy the Codazzi-Gauss conditions of surface theory:

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right)=\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2},\\
&\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right)=\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1},\\
&\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right)+\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right)=-\frac{A_1A_2}{R_1R_2}.
\end{aligned}
\tag{18}
$$

Substituting Eq. (17) into Eq. (10) and using Eq. (18), the rotation vector components are obtained as

$$
\begin{aligned}
&\omega_1=\frac{1}{2}\left[\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}-\frac{u_2}{R_2}\right)-\frac{\partial u_2}{\partial z}\right],\\
&\omega_2=\frac{1}{2}\left[\frac{\partial u_1}{\partial z}-\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}-\frac{u_1}{R_1}\right)\right],\\
&\omega_3=\frac{1}{2}\left[\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)-\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)\right].
\end{aligned}
\tag{19}
$$

The rotation gradient tensor components can be obtained in a similar fashion as

$$
\begin{aligned}
&\chi_{11}=\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial\omega_1}{\partial\alpha_1}+\frac{\omega_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{\omega_3}{R_1}\right),\\
&\chi_{22}=\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial\omega_2}{\partial\alpha_2}+\frac{\omega_1}{A_2A_1}\frac{\partial A_2}{\partial\alpha_1}+\frac{\omega_3}{R_2}\right),\\
&\chi_{33}=\frac{\partial\omega_3}{\partial z},\\
&\chi_{12}=\frac{1}{2}\left[\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial\omega_2}{\partial\alpha_1}-\frac{\omega_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial\omega_1}{\partial\alpha_2}-\frac{\omega_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)\right],\\
&\chi_{13}=\frac{1}{2}\left[\frac{\partial\omega_1}{\partial z}+\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial\omega_3}{\partial\alpha_1}-\frac{\omega_1}{R_1}\right)\right],\\
&\chi_{23}=\frac{1}{2}\left[\frac{\partial\omega_2}{\partial z}+\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial\omega_3}{\partial\alpha_2}-\frac{\omega_2}{R_2}\right)\right],
\end{aligned}
\tag{20}
$$

which can be rewritten in terms of the components of the displacement vector as

$$\chi_{11}=\frac{1}{2}\frac{1}{\left(1+z/R_1\right)}\left\{\frac{1}{A_2\left(1+z/R_2\right)}\left[\frac{1}{A_1}\frac{\partial^2 u_3}{\partial\alpha_1\partial\alpha_2}-\frac{1}{A_1}\frac{\left(1+z/R_1\right)}{A_2\left(1+z/R_2\right)}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial u_3}{\partial\alpha_2}\right]-\frac{1}{A_1}\frac{\partial^2 u_2}{\partial\alpha_1\partial z}\right.$$
$$-\frac{1}{R_2\left(1+z/R_2\right)}\left[\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{A_1}\frac{u_2}{R_2\left(1+z/R_2\right)}\frac{\partial R_2}{\partial\alpha_1}\right]+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\left[\frac{\partial u_1}{\partial z}-\frac{1}{A_1\left(1+z/R_1\right)}\frac{\partial u_3}{\partial\alpha_1}\right]$$
$$\left.+\frac{1}{R_1}\left[\frac{1}{A_1\left(1+z/R_1\right)}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)\right]\right\},$$

$$\chi_{22}=\frac{1}{2}\frac{1}{\left(1+z/R_2\right)}\left\{\frac{1}{A_2}\frac{\partial^2 u_1}{\partial\alpha_2\partial z}-\frac{1}{A_1\left(1+z/R_1\right)}\left[\frac{1}{A_2}\frac{\partial^2 u_3}{\partial\alpha_2\partial\alpha_1}-\frac{1}{A_2}\frac{\left(1+z/R_2\right)}{A_1\left(1+z/R_1\right)}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial u_3}{\partial\alpha_1}\right]\right.$$
$$+\frac{1}{R_1\left(1+z/R_1\right)}\left[\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{u_1}{R_1\left(1+z/R_1\right)}\frac{1}{A_2}\frac{\partial R_1}{\partial\alpha_2}\right]+\frac{1}{A_2A_1}\frac{\partial A_2}{\partial\alpha_1}\left[\frac{1}{A_2\left(1+z/R_2\right)}\frac{\partial u_3}{\partial\alpha_2}-\frac{\partial u_2}{\partial z}\right]$$
$$\left.+\frac{1}{R_2}\left[\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)-\frac{1}{A_2\left(1+z/R_2\right)}\frac{\partial u_1}{\partial\alpha_2}\right]\right\},$$

$$\chi_{33}=\frac{1}{2}\left\{\frac{1}{A_1\left(1+z/R_1\right)}\left[\left(\frac{\partial^2 u_2}{\partial\alpha_1\partial z}-\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial u_1}{\partial z}\right)-\frac{1}{R_1\left(1+z/R_1\right)}\left(\frac{\partial u_2}{\partial\alpha_1}-\frac{u_1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right)\right]\right.$$
$$\left.-\frac{1}{A_2\left(1+z/R_2\right)}\left[\left(\frac{\partial^2 u_1}{\partial\alpha_2\partial z}-\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial u_2}{\partial z}\right)-\frac{1}{R_2\left(1+z/R_2\right)}\left(\frac{\partial u_1}{\partial\alpha_2}-\frac{u_2}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right)\right]\right\},\qquad (21)$$

$$
\begin{aligned}
\chi_{12} = \frac{1}{4}\Bigg\{ & \frac{1}{A_2\left(1+z/R_2\right)}\Bigg[\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2\left(1+z/R_2\right)}\right)\frac{\partial u_3}{\partial\alpha_2}+\frac{1}{A_2\left(1+z/R_2\right)}\frac{\partial^2 u_3}{\partial\alpha_2^2}-\frac{\partial^2 u_2}{\partial\alpha_2\partial z}-\frac{1}{R_2\left(1+z/R_2\right)}\frac{\partial u_2}{\partial\alpha_2} \\
& +\frac{u_2}{\left(R_2\left(1+z/R_2\right)\right)^2}\frac{\partial R_2}{\partial\alpha_2}-\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{\partial u_1}{\partial z}-\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}-\frac{u_1}{R_1}\right)\right)\Bigg] \\
& +\frac{1}{A_1\left(1+z/R_1\right)}\Bigg[\frac{\partial^2 u_1}{\partial\alpha_1\partial z}-\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1\left(1+z/R_1\right)}\right)\frac{\partial u_3}{\partial\alpha_1}-\frac{1}{A_1\left(1+z/R_1\right)}\frac{\partial^2 u_3}{\partial\alpha_1^2}+\frac{1}{R_1\left(1+z/R_1\right)}\frac{\partial u_1}{\partial\alpha_1} \\
& -\frac{u_1}{\left(R_1\left(1+z/R_1\right)\right)^2}\frac{\partial R_1}{\partial\alpha_1}-\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}-\frac{u_2}{R_2}\right)-\frac{\partial u_2}{\partial z}\right)\Bigg]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{13} = \frac{1}{4}\Bigg\{ & \Bigg[\frac{1}{A_2\left(1+z/R_2\right)}\left(\frac{\partial^2 u_3}{\partial\alpha_2\partial z}-\frac{1}{R_2\left(1+z/R_2\right)}\frac{\partial u_3}{\partial\alpha_2}\right)-\frac{\partial^2 u_2}{\partial z^2}-\frac{1}{R_2\left(1+z/R_2\right)}\left(\frac{\partial u_2}{\partial z}-\frac{u_2}{R_2\left(1+z/R_2\right)}\right)\Bigg] \\
& +\frac{1}{A_1\left(1+z/R_1\right)}\Bigg[\frac{1}{A_1\left(1+z/R_1\right)}\left(\frac{\partial^2 u_2}{\partial\alpha_1^2}-\frac{u_1}{A_2}\frac{\partial^2 A_1}{\partial\alpha_1\partial\alpha_2}-\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_1}\frac{\partial A_1}{\partial\alpha_2}+\frac{u_1}{A_2^2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial A_1}{\partial\alpha_2}\right) \\
& +\frac{1}{A_2\left(1+z/R_2\right)}\left(-\frac{\partial^2 u_1}{\partial\alpha_1\partial\alpha_2}+\frac{u_2}{A_1}\frac{\partial^2 A_2}{\partial\alpha_1^2}+\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}\frac{\partial A_2}{\partial\alpha_1}-\frac{u_2}{A_1^2}\frac{\partial A_1}{\partial\alpha_1}\frac{\partial A_2}{\partial\alpha_1}\right) \\
& +\frac{\left(1+z/R_1\right)}{\left(A_2\left(1+z/R_2\right)\right)^2}\left(\frac{\partial A_2}{\partial\alpha_1}\frac{\partial u_1}{\partial\alpha_2}-\frac{u_2}{A_1}\left(\frac{\partial A_2}{\partial\alpha_1}\right)^2\right)+\left(\frac{\partial u_2}{\partial\alpha_1}-\frac{u_1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right)\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1\left(1+z/R_1\right)}\right)\Bigg] \\
& -\frac{1}{R_1\left(1+z/R_1\right)}\Bigg[\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}-\frac{u_2}{R_2}\right)-\frac{\partial u_2}{\partial z}\Bigg]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{23} = \frac{1}{4}\Bigg\{ & \Bigg[\frac{\partial^2 u_1}{\partial z^2}-\frac{1}{A_1\left(1+z/R_1\right)}\left(\frac{\partial^2 u_3}{\partial\alpha_1\partial z}-\frac{1}{R_1\left(1+z/R_1\right)}\frac{\partial u_3}{\partial\alpha_1}\right)+\frac{1}{R_1\left(1+z/R_1\right)}\frac{\partial u_1}{\partial z}-\frac{u_1}{\left(R_1\left(1+z/R_1\right)\right)^2}\Bigg] \\
& +\frac{1}{A_2\left(1+z/R_2\right)}\Bigg[\frac{1}{A_1\left(1+z/R_1\right)}\left(\frac{\partial^2 u_2}{\partial\alpha_1\partial\alpha_2}-\frac{u_1}{A_2}\frac{\partial^2 A_1}{\partial\alpha_2^2}-\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{u_1}{A_2^2}\frac{\partial A_2}{\partial\alpha_2}\frac{\partial A_1}{\partial\alpha_2}\right) \\
& +\frac{1}{A_2\left(1+z/R_2\right)}\left(-\frac{\partial^2 u_1}{\partial\alpha_2^2}+\frac{u_2}{A_1}\frac{\partial^2 A_2}{\partial\alpha_1\partial\alpha_2}+\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_2}\frac{\partial A_2}{\partial\alpha_1}-\frac{u_2}{A_1^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial A_2}{\partial\alpha_1}\right) \\
& +\left(\frac{u_2}{A_1}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial u_1}{\partial\alpha_2}\right)\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2\left(1+z/R_2\right)}\right)+\frac{\left(1+z/R_2\right)}{\left(A_1\left(1+z/R_1\right)\right)^2}\left(\frac{u_1}{A_2}\left(\frac{\partial A_1}{\partial\alpha_2}\right)^2-\frac{\partial A_1}{\partial\alpha_2}\frac{\partial u_2}{\partial\alpha_1}\right)\Bigg] \\
& -\frac{1}{R_2\left(1+z/R_2\right)}\Bigg[\frac{\partial u_1}{\partial z}-\frac{1}{A_1\left(1+z/R_1\right)}\frac{\partial u_3}{\partial\alpha_1}+\frac{u_1}{R_1\left(1+z/R_1\right)}\Bigg]\Bigg\}.
\end{aligned}
$$

The nonlinear Green strain tensor components for the doubly curved coordinate system under consideration can be obtained by substituting Eq. (17) into Eq. (16) and making use of sing Eq. (18), resulting in

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1}+\frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}+\frac{u_3}{R_1}\right)+\frac{1}{2}\frac{1}{\left(1+z/R_1\right)^2}\left[\left(\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1}+\frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}+\frac{u_3}{R_1}\right)^2\right.\\
&\quad \left.+\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1}-\frac{u_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)^2+\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}-\frac{u_1}{R_1}\right)^2\right],\\[2ex]
\varepsilon_{22} &= \frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2}+\frac{u_1}{A_2 A_1}\frac{\partial A_2}{\partial \alpha_1}+\frac{u_3}{R_2}\right)+\frac{1}{2}\frac{1}{\left(1+z/R_2\right)^2}\left[\left(\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2}+\frac{u_1}{A_2 A_1}\frac{\partial A_2}{\partial \alpha_1}+\frac{u_3}{R_2}\right)^2\right.\\
&\quad \left.+\left(\frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2}-\frac{u_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\right)^2+\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}-\frac{u_2}{R_2}\right)^2\right],\\[2ex]
\varepsilon_{33} &= \frac{\partial u_3}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u_3}{\partial z}\right)^2+\left(\frac{\partial u_1}{\partial z}\right)^2+\left(\frac{\partial u_2}{\partial z}\right)^2\right],\\[2ex]
\varepsilon_{12} &= \frac{1}{2}\left[\frac{1}{A_1\left(1+z/R_1\right)}\left(\frac{\partial u_2}{\partial \alpha_1}-\frac{u_1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\right)+\frac{1}{A_2\left(1+z/R_2\right)}\left(\frac{\partial u_1}{\partial \alpha_2}-\frac{u_2}{A_1}\frac{\partial A_2}{\partial \alpha_1}\right)\right]\\
&\quad +\frac{1}{2}\frac{1}{\left(1+z/R_1\right)}\frac{1}{\left(1+z/R_2\right)}\left[\left(\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1}+\frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}+\frac{u_3}{R_1}\right)\left(\frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2}-\frac{u_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\right)\right.\\
&\quad +\left(\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2}+\frac{u_1}{A_2 A_1}\frac{\partial A_2}{\partial \alpha_1}+\frac{u_3}{R_2}\right)\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1}-\frac{u_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\\
&\quad \left.+\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}-\frac{u_1}{R_1}\right)\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}-\frac{u_2}{R_2}\right)\right],\\[2ex]
\varepsilon_{13} &= \frac{1}{2}\frac{\partial u_1}{\partial z}+\frac{1}{2}\frac{1}{\left(1+z/R_1\right)}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}-\frac{u_1}{R_1}\right)+\frac{1}{2}\frac{1}{\left(1+z/R_1\right)}\left[\frac{\partial u_1}{\partial z}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1}+\frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}+\frac{u_3}{R_1}\right)\right.\\
&\quad \left.+\frac{\partial u_2}{\partial z}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1}-\frac{u_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)+\frac{\partial u_3}{\partial z}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}-\frac{u_1}{R_1}\right)\right],
\end{aligned} \tag{22}$$

$$\varepsilon_{23}=\frac{1}{2}\frac{\partial u_2}{\partial z}+\frac{1}{2}\frac{1}{\left(1+z/R_2\right)}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}-\frac{u_2}{R_2}\right)+\frac{1}{2}\frac{1}{\left(1+z/R_2\right)}\left[\frac{\partial u_1}{\partial z}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2}-\frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\right)\right.$$
$$\left.+\frac{\partial u_2}{\partial z}\left(\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2}+\frac{u_1}{A_2A_1}\frac{\partial A_2}{\partial \alpha_1}+\frac{u_3}{R_2}\right)+\frac{\partial u_3}{\partial z}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}-\frac{u_2}{R_2}\right)\right].$$

5. Modified couple stress theory formulations for special cases

The MCST formulations for coordinate systems which are special cases of the doubly curved coordinate system derived in Section 4 are presented in this section. In particular, the MCST formulations are presented in cylindrical and spherical coordinates, which are specifically useful in problems that require modelling of cylindrical or spherical shells. It should be mentioned here again that for the doubly curved coordinate system defined in Section 4, *z* denotes the radial coordinate, which is perpendicular to the angular coordinates α_1 and α_2. Hence, in the following, the cylindrical and spherical radial coordinate is denoted by *z*. This notation is chosen for consistency with shell theories formulations.

5.1 MCST formulations in cylindrical coordinates

In this section, the general MCST formulations of Section 4 are reduced to the case of a cylindrical coordinate system, which is particularly useful for modelling of cylindrical shells. For such a coordinate system

$$R_1\to\infty,\ A_1=1,\alpha_1=x,$$
$$R_2=R,\ A_2=R,\ \alpha_2=\theta, \tag{23}$$

in which R is the radius of the cylinder; x is the longitudinal axis of the cylinder, while z and θ are the polar coordinates. Again, as mentioned in Section 4, z is the radial coordinate and perpendicular to x and θ coordinate lines.

Hence, the rotation vector components become

$$\begin{aligned}
\omega_1 &= \frac{1}{2}\left[\frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial u_3}{\partial \theta} - \frac{u_2}{R}\right) - \frac{\partial u_2}{\partial z}\right], \\
\omega_2 &= \frac{1}{2}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right), \\
\omega_3 &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x} - \frac{1}{R(1+z/R)}\frac{\partial u_1}{\partial \theta}\right).
\end{aligned} \tag{24}$$

The rotation gradient tensor components, in terms of rotation vector components, reduce to

$$\begin{aligned}
\chi_{xx} &= \frac{\partial \omega_1}{\partial x}, \\
\chi_{\theta\theta} &= \frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial \omega_2}{\partial \theta} + \frac{\omega_3}{R}\right), \\
\chi_{zz} &= \frac{\partial \omega_3}{\partial z}, \\
\chi_{x\theta} &= \frac{1}{2}\left(\frac{1}{R(1+z/R)}\frac{\partial \omega_1}{\partial \theta} + \frac{\partial \omega_2}{\partial x}\right), \\
\chi_{xz} &= \frac{1}{2}\left(\frac{\partial \omega_1}{\partial z} + \frac{\partial \omega_3}{\partial x}\right), \\
\chi_{\theta z} &= \frac{1}{2}\left[\frac{\partial \omega_2}{\partial z} + \frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial \omega_3}{\partial \theta} - \frac{\omega_2}{R}\right)\right],
\end{aligned} \tag{25}$$

which can be rewritten in terms of the components of the displacement vector as

$$\chi_{xx}=\frac{1}{2}\left[\frac{1}{R(1+z/R)}\left(\frac{\partial^2 u_3}{\partial x\partial\theta}-\frac{\partial u_2}{\partial x}\right)-\frac{\partial^2 u_2}{\partial x\partial z}\right],$$

$$\chi_{\theta\theta}=\frac{1}{2}\frac{1}{R(1+z/R)}\left[\left(\frac{\partial^2 u_1}{\partial\theta\partial z}-\frac{\partial^2 u_3}{\partial x\partial\theta}\right)+\left(\frac{\partial u_2}{\partial x}-\frac{1}{R(1+z/R)}\frac{\partial u_1}{\partial\theta}\right)\right],$$

$$\chi_{zz}=\frac{1}{2}\left[\frac{\partial^2 u_2}{\partial x\partial z}-\frac{1}{R(1+z/R)}\left(\frac{\partial^2 u_1}{\partial\theta\partial z}-\frac{1}{R(1+z/R)}\frac{\partial u_1}{\partial\theta}\right)\right],$$

$$\chi_{x\theta}=\frac{1}{4}\left[\frac{\partial^2 u_1}{\partial x\partial z}-\frac{\partial^2 u_3}{\partial x^2}-\frac{1}{R(1+z/R)}\frac{\partial^2 u_2}{\partial\theta\partial z}+\frac{1}{\left(R(1+z/R)\right)^2}\left(\frac{\partial^2 u_3}{\partial\theta^2}-\frac{\partial u_2}{\partial\theta}\right)\right],$$

$$\chi_{xz}=\frac{1}{4}\left[\frac{\partial^2 u_2}{\partial x^2}-\frac{\partial^2 u_2}{\partial z^2}+\frac{1}{R(1+z/R)}\left(\frac{\partial^2 u_3}{\partial\theta\partial z}-\frac{\partial^2 u_1}{\partial x\partial\theta}-\frac{\partial u_2}{\partial z}\right)+\frac{1}{\left(R(1+z/R)\right)^2}\left(u_2-\frac{\partial u_3}{\partial\theta}\right)\right],$$

$$\chi_{\theta z}=\frac{1}{4}\left[\frac{\partial^2 u_1}{\partial z^2}-\frac{\partial^2 u_3}{\partial x\partial z}+\frac{1}{R(1+z/R)}\left(\frac{\partial^2 u_2}{\partial x\partial\theta}-\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)-\frac{1}{\left(R(1+z/R)\right)^2}\frac{\partial^2 u_1}{\partial\theta^2}\right].\tag{26}$$

The nonlinear Green strain tensor components in a cylindrical coordinate can be obtained as

$$\varepsilon_{xx}=\frac{\partial u_1}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2+\left(\frac{\partial u_3}{\partial x}\right)^2\right],$$

$$\begin{aligned}\varepsilon_{\theta\theta}=\frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial u_2}{\partial\theta}+\frac{u_3}{R}\right)+\frac{1}{2}\frac{1}{(1+z/R)^2}\Bigg[&\left(\frac{1}{R}\frac{\partial u_2}{\partial\theta}+\frac{u_3}{R}\right)^2\\&+\left(\frac{1}{R}\frac{\partial u_1}{\partial\theta}\right)^2+\left(\frac{1}{R}\frac{\partial u_3}{\partial\theta}-\frac{u_2}{R}\right)^2\Bigg],\end{aligned}$$

$$\varepsilon_{zz}=\frac{\partial u_3}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u_3}{\partial z}\right)^2+\left(\frac{\partial u_1}{\partial z}\right)^2+\left(\frac{\partial u_2}{\partial z}\right)^2\right],\tag{27}$$

$$\begin{aligned}\varepsilon_{x\theta}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x}+\frac{1}{R(1+z/R)}\frac{\partial u_1}{\partial\theta}\right)+\frac{1}{2}\frac{1}{(1+z/R)}\Bigg[&\frac{1}{R}\frac{\partial u_1}{\partial\theta}\frac{\partial u_1}{\partial x}\\&+\left(\frac{1}{R}\frac{\partial u_2}{\partial\theta}+\frac{u_3}{R}\right)\frac{\partial u_2}{\partial x}+\frac{\partial u_3}{\partial x}\left(\frac{1}{R}\frac{\partial u_3}{\partial\theta}-\frac{u_2}{R}\right)\Bigg],\end{aligned}$$

$$\varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)+\frac{1}{2}\left[\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial z}+\frac{\partial u_2}{\partial x}\frac{\partial u_2}{\partial z}+\frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial z}\right],$$

$$\varepsilon_{\theta z}=\frac{1}{2}\frac{\partial u_2}{\partial z}+\frac{1}{2}\left(\frac{1}{R\left(1+z/R\right)}\frac{\partial u_3}{\partial \theta}-\frac{u_2}{R\left(1+z/R\right)}\right)+\frac{1}{2}\frac{1}{\left(1+z/R\right)}\left[\frac{\partial u_2}{\partial z}\left(\frac{1}{R}\frac{\partial u_2}{\partial \theta}+\frac{u_3}{R}\right)\right.$$
$$\left.+\frac{\partial u_3}{\partial z}\left(\frac{1}{R}\frac{\partial u_3}{\partial \theta}-\frac{u_2}{R}\right)+\frac{1}{R}\frac{\partial u_1}{\partial \theta}\frac{\partial u_1}{\partial z}\right].$$

5.2 MCST formulations in spherical coordinates

In this section, a procedure similar to that of Section 5.1 is followed but this time for a spherical coordinate system shown in Fig. 2. For such a coordinate system

$$\begin{aligned}&R_1=R,\ A_1=R,\ \alpha_1=\varphi\\&R_2=R,\ A_2=R\sin\varphi,\ \alpha_2=\theta.\end{aligned}\tag{28}$$

The rotation vector components in the spherical coordinate system become

$$\begin{aligned}\omega_1&=\frac{1}{2}\left[\frac{1}{\left(1+z/R\right)}\left(\frac{1}{R\sin\varphi}\frac{\partial u_3}{\partial \theta}-\frac{u_2}{R}\right)-\frac{\partial u_2}{\partial z}\right],\\\omega_2&=\frac{1}{2}\left[\frac{\partial u_1}{\partial z}-\frac{1}{\left(1+z/R\right)}\left(\frac{1}{R}\frac{\partial u_3}{\partial \varphi}-\frac{u_1}{R}\right)\right],\\\omega_3&=\frac{1}{2}\frac{1}{\left(1+z/R\right)}\left[\frac{1}{R}\frac{\partial u_2}{\partial \varphi}-\frac{1}{R\sin\varphi}\frac{\partial u_1}{\partial \theta}+\frac{u_2}{R\tan\varphi}\right].\end{aligned}\tag{29}$$

Then, one can obtain the rotation gradient tensor components in terms of rotation vector components as

$$
\begin{aligned}
&\chi_{\varphi\varphi} = \frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial \omega_1}{\partial \varphi} + \frac{\omega_3}{R}\right), \\
&\chi_{\theta\theta} = \frac{1}{(1+z/R)}\left(\frac{1}{R\sin\varphi}\frac{\partial \omega_2}{\partial \theta} + \frac{\omega_1}{R\tan\varphi} + \frac{\omega_3}{R}\right), \\
&\chi_{zz} = \frac{\partial \omega_3}{\partial z}, \\
&\chi_{\varphi\theta} = \frac{1}{2}\frac{1}{(1+z/R)}\left[\frac{1}{R\sin\varphi}\frac{\partial \omega_1}{\partial \theta} + \frac{1}{R}\frac{\partial \omega_2}{\partial \varphi} - \frac{\omega_2}{R\tan\varphi}\right] \\
&\chi_{\varphi z} = \frac{1}{2}\left[\frac{\partial \omega_1}{\partial z} + \frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial \omega_3}{\partial \varphi} - \frac{\omega_1}{R}\right)\right], \\
&\chi_{\theta z} = \frac{1}{2}\left[\frac{\partial \omega_2}{\partial z} + \frac{1}{(1+z/R)}\left(\frac{1}{R\sin\varphi}\frac{\partial \omega_3}{\partial \theta} - \frac{\omega_2}{R}\right)\right],
\end{aligned}
\tag{30}
$$

which can be expanded to

$$
\begin{aligned}
\chi_{\varphi\varphi} = \frac{1}{2}\frac{1}{(1+z/R)}\Bigg\{&\frac{1}{R(1+z/R)}\left[\frac{1}{R\sin\varphi}\frac{\partial^2 u_3}{\partial\varphi\partial\theta} - \frac{1}{R\tan\varphi\sin\varphi}\frac{\partial u_3}{\partial \theta}\right] - \frac{1}{R}\frac{\partial^2 u_2}{\partial\varphi\partial z} \\
&+ \frac{1}{R(1+z/R)}\left(\frac{u_2}{R\tan\varphi} - \frac{1}{R\sin\varphi}\frac{\partial u_1}{\partial \theta}\right)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{\theta\theta} = \frac{1}{2}\frac{1}{(1+z/R)}\Bigg\{&\frac{1}{R(1+z/R)}\frac{1}{R}\frac{\partial u_2}{\partial \varphi} + \frac{1}{R\sin\varphi}\left[\frac{\partial^2 u_1}{\partial\theta\partial z} - \frac{1}{R(1+z/R)}\frac{\partial^2 u_3}{\partial\theta\partial\varphi}\right] \\
&+ \frac{1}{R\tan\varphi}\left[\frac{1}{R\sin\varphi}\frac{1}{(1+z/R)}\frac{\partial u_3}{\partial \theta} - \frac{\partial u_2}{\partial z}\right]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{zz} = \frac{1}{2}\Bigg[&\frac{1}{R(1+z/R)}\frac{\partial^2 u_2}{\partial\varphi\partial z} - \frac{1}{\left(R(1+z/R)\right)^2}\left(\frac{\partial u_2}{\partial \varphi} + \frac{u_2}{\tan\varphi} - \frac{1}{\sin\varphi}\frac{\partial u_1}{\partial \theta}\right) \\
&- \frac{1}{R\sin\varphi(1+z/R)}\left(\frac{\partial^2 u_1}{\partial\theta\partial z} - \frac{\partial u_2}{\partial z}\cos\varphi\right)\Bigg],
\end{aligned}
\tag{31}
$$

$$\chi_{\varphi\theta}=\frac{1}{4}\left\{\frac{1}{R\sin\varphi(1+z/R)}\left[\frac{1}{(1+z/R)}\left(\frac{1}{R\sin\varphi}\frac{\partial^2 u_3}{\partial\theta^2}-\frac{1}{R}\frac{\partial u_2}{\partial\theta}\right)-\frac{\partial^2 u_2}{\partial\theta\partial z}\right]\right.$$
$$+\frac{1}{R(1+z/R)}\left[\frac{\partial^2 u_1}{\partial\varphi\partial z}-\frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial^2 u_3}{\partial\varphi^2}-\frac{1}{R}\frac{\partial u_1}{\partial\varphi}\right)\right]$$
$$\left.-\frac{1}{R\tan\varphi}\frac{1}{(1+z/R)}\left[\frac{\partial u_1}{\partial z}-\frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial u_3}{\partial\varphi}-\frac{u_1}{R}\right)\right]\right\},$$

$$\chi_{\varphi z}=\frac{1}{4}\left\{\left[\frac{1}{(1+z/R)}\left(\frac{1}{R\sin\varphi}\frac{\partial^2 u_3}{\partial\theta\partial z}\right)+\frac{1}{\left(R(1+z/R)\right)^2}\left(u_2-\frac{2}{\sin\varphi}\frac{\partial u_3}{\partial\theta}\right)-\frac{\partial^2 u_2}{\partial z^2}\right]\right.$$
$$\left.+\frac{1}{\left(R(1+z/R)\right)^2}\left[\frac{\partial^2 u_2}{\partial\varphi^2}-\frac{1}{\sin\varphi}\frac{\partial^2 u_1}{\partial\varphi\partial\theta}+\frac{1}{\sin\varphi\tan\varphi}\frac{\partial u_1}{\partial\theta}+\frac{1}{\tan\varphi}\frac{\partial u_2}{\partial\varphi}-\frac{u_2}{(\tan\varphi)^2}\right]\right\},$$

$$\chi_{\theta z}=\frac{1}{4}\left[\frac{\partial^2 u_1}{\partial z^2}-\frac{1}{R(1+z/R)}\frac{\partial^2 u_3}{\partial\varphi\partial z}+\frac{2}{\left(R(1+z/R)\right)^2}\left(\frac{\partial u_3}{\partial\varphi}-u_1\right)\right.$$
$$\left.+\frac{1}{\sin\varphi\left(R(1+z/R)\right)^2}\left(\frac{\partial^2 u_2}{\partial\varphi\partial\theta}-\frac{1}{\sin\varphi}\frac{\partial^2 u_1}{\partial\theta^2}+\frac{1}{\tan\varphi}\frac{\partial u_2}{\partial\theta}\right)\right].$$

Finally, the nonlinear strain components in the spherical coordinate system under consideration can be expressed as

$$\varepsilon_{\varphi\varphi}=\frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial u_1}{\partial\varphi}+\frac{u_3}{R}\right)+\frac{1}{2}\frac{1}{(1+z/R)^2}\left[\left(\frac{1}{R}\frac{\partial u_1}{\partial\varphi}+\frac{u_3}{R}\right)^2+\left(\frac{1}{R}\frac{\partial u_2}{\partial\varphi}\right)^2+\left(\frac{1}{R}\frac{\partial u_3}{\partial\varphi}-\frac{u_1}{R}\right)^2\right],$$

$$\varepsilon_{\theta\theta}=\frac{1}{(1+z/R)}\left(\frac{1}{R\sin\varphi}\frac{\partial u_2}{\partial\theta}+\frac{u_1}{R\tan\varphi}+\frac{u_3}{R}\right)+\frac{1}{2}\frac{1}{(1+z/R)^2}\left[\left(\frac{1}{R\sin\varphi}\frac{\partial u_2}{\partial\theta}+\frac{u_1}{R\tan\varphi}+\frac{u_3}{R}\right)^2\right.$$
$$\left.+\left(\frac{1}{R\sin\varphi}\frac{\partial u_1}{\partial\theta}-\frac{u_2}{R\tan\varphi}\right)^2+\left(\frac{1}{R\sin\varphi}\frac{\partial u_3}{\partial\theta}-\frac{u_2}{R}\right)^2\right],$$

$$\varepsilon_{zz}=\frac{\partial u_3}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u_3}{\partial z}\right)^2+\left(\frac{\partial u_1}{\partial z}\right)^2+\left(\frac{\partial u_2}{\partial z}\right)^2\right],\tag{32}$$

$$\varepsilon_{\varphi\theta}=\frac{1}{2}\left[\frac{1}{R(1+z/R)}\left(\frac{\partial u_2}{\partial\varphi}\right)+\frac{1}{R\sin\varphi(1+z/R)}\left(\frac{\partial u_1}{\partial\theta}-u_2\cos\varphi\right)\right]$$
$$+\frac{1}{2}\frac{1}{(1+z/R)^2}\left[\left(\frac{1}{R}\frac{\partial u_1}{\partial\varphi}+\frac{u_3}{R}\right)\left(\frac{1}{R\sin\varphi}\frac{\partial u_1}{\partial\theta}-\frac{u_2}{R\tan\varphi}\right)\right.$$
$$\left.+\frac{1}{R}\frac{\partial u_2}{\partial\varphi}\left(\frac{1}{R\sin\varphi}\frac{\partial u_2}{\partial\theta}+\frac{u_1}{R\tan\varphi}+\frac{u_3}{R}\right)+\left(\frac{1}{R}\frac{\partial u_3}{\partial\varphi}-\frac{u_1}{R}\right)\left(\frac{1}{R\sin\varphi}\frac{\partial u_3}{\partial\theta}-\frac{u_2}{R}\right)\right],$$

$$\varepsilon_{\varphi z}=\frac{1}{2}\frac{\partial u_1}{\partial z}+\frac{1}{2}\frac{1}{(1+z/R)}\left(\frac{1}{R}\frac{\partial u_3}{\partial\varphi}-\frac{u_1}{R}\right)+\frac{1}{2}\frac{1}{(1+z/R)}\left[\frac{\partial u_1}{\partial z}\left(\frac{1}{R}\frac{\partial u_1}{\partial\varphi}+\frac{u_3}{R}\right)\right.$$
$$\left.+\frac{\partial u_2}{\partial z}\left(\frac{1}{R}\frac{\partial u_2}{\partial\varphi}\right)+\frac{\partial u_3}{\partial z}\left(\frac{1}{R}\frac{\partial u_3}{\partial\varphi}-\frac{u_1}{R}\right)\right],$$

$$\varepsilon_{\theta z}=\frac{1}{2}\frac{\partial u_2}{\partial z}+\frac{1}{2}\frac{1}{(1+z/R)}\left(\frac{1}{R\sin\varphi}\frac{\partial u_3}{\partial\theta}-\frac{u_2}{R}\right)+\frac{1}{2}\frac{1}{(1+z/R)}\left[\frac{\partial u_1}{\partial z}\left(\frac{1}{R\sin\varphi}\frac{\partial u_1}{\partial\theta}-\frac{u_2}{R\tan\varphi}\right)\right.$$
$$\left.+\frac{\partial u_2}{\partial z}\left(\frac{1}{R\sin\varphi}\frac{\partial u_2}{\partial\theta}+\frac{u_1}{R\tan\varphi}+\frac{u_3}{R}\right)+\frac{\partial u_3}{\partial z}\left(\frac{1}{R\sin\varphi}\frac{\partial u_3}{\partial\theta}-\frac{u_2}{R}\right)\right].$$

6. Equations of motion and numerical simulations for the case of spherical microshells

In this section, equations of motion of a spherical microshell are obtained and numerical simulations are conducted to calculate the linear natural frequencies as well as the nonlinear static bending. To this end, the kinetic and potential energies of the system are constructed and substituted into Lagrange's equations to obtain the discretised equations of motion. The linear natural frequencies are calculated via an eigenvalue analysis while the nonlinear bending characteristics are obtained making use of a continuation technique.

The components of the strain tensor $\boldsymbol{\varepsilon}$ and the symmetric rotation gradient tensor $\boldsymbol{\chi}$ have already been obtained in Section 5.2. Under plane-stress condition, the components of $\boldsymbol{\sigma}$ can be written in terms of those of $\boldsymbol{\varepsilon}$ as

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right), \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right), \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}.
\end{aligned} \tag{33}$$

For the case of higher-order stress and strain tensors (i.e. $\mathbf{m}$ and $\boldsymbol{\chi}$) this relation is given by

$$m_{ij} = 2\mu l^2 \chi_{ij}, \tag{34}$$

For the case of a shallow spherical microshell, the following simplifications can be made

$$\sin\varphi \approx 1, \qquad \frac{1}{\tan\varphi} \approx 0, \qquad \cos\varphi \approx 0. \tag{35}$$

Additionally, the notations $x = R\varphi$ and $y = R\theta$ are used for brevity, knowing that the microshell dimensions are a, b, and h in the x, y, and z directions, respectively.

The potential energy of the spherical microshell can then be constructed as

$$\begin{aligned}
\Pi &= \frac{1}{2}\int_V \left(\boldsymbol{\sigma}:\boldsymbol{\varepsilon} + \mathbf{m}:\boldsymbol{\chi}\right)\left(1+\frac{z}{R}\right)^2 \mathrm{d}v \\
&= \frac{1}{2}\int_0^a\int_0^b\int_{-\frac{h}{2}}^{\frac{h}{2}} \Big\{\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy}\right) \\
&\quad + \left[m_{xx}\chi_{xx} + m_{yy}\chi_{yy} + m_{zz}\chi_{zz} + 2\left(m_{xy}\chi_{xy} + m_{xz}\chi_{xz} + m_{yz}\chi_{yz}\right)\right]\Big\}\left(1+\frac{z}{R}\right)^2 \mathrm{d}z\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{36}$$

The kinetic energy of the spherical microshell can be formulated as

$$T = \frac{1}{2}\rho\int_0^a\int_0^b\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right]\left(1+\frac{z}{R}\right)^2 \mathrm{d}z\mathrm{d}y\mathrm{d}x, \tag{37}$$

in which

$$
\begin{aligned}
u_1 &= \left(1+\frac{z}{R}\right)u - z\frac{\partial w}{\partial x}, \\
u_2 &= \left(1+\frac{z}{R}\right)v - z\frac{\partial w}{\partial y}, \qquad (38) \\
u_3 &= w,
\end{aligned}
$$

where *u* and *v* are the mid-plane displacements in the *x* and *y* directions (in-plane) and *w* is the mid-plane displacement in the *z* direction (out-of-plane).

A static pressure load *p* is applied to the spherical microshell in the *positive z* direction; the work of this load can be formulated as

$$
W_p = \int_0^a\int_0^b p\,w\left(x,y,t\right)\mathrm{d}y\mathrm{d}x. \qquad (39)
$$

To derive the discretised equations of motion via Lagrange's equations, an assumed-mode technique is used. For a fully-clamped microshell, the continuous mid-plane displacements are discretised to a series of time-dependent terms multiplied by trial functions of *x* and *y* as

$$
\begin{aligned}
u(x,y,t) &= \sum_{m=1}^{M_u}\sum_{n=1}^{N_u} u_{m,n}\left(t\right)\Phi_m\left(x/a\right)\Phi_n\left(y/b\right), \\
v(x,y,t) &= \sum_{m=1}^{M_v}\sum_{n=1}^{N_v} v_{m,n}\left(t\right)\Phi_m\left(x/a\right)\Phi_n\left(y/b\right), \qquad (40) \\
w(x,y,t) &= \sum_{m=1}^{M_w}\sum_{n=1}^{N_w} w_{m,n}\left(t\right)\Lambda_m\left(x/a\right)\Lambda_n\left(y/b\right),
\end{aligned}
$$

where the time-dependent terms are shown by $u_{m,n}(t)$, $v_{m,n}(t)$, and $w_{m,n}(t)$ and the trial functions are denoted by Φ and Λ given by

$$
\Phi_m\left(x/a\right) = \sin\left(m\pi x/a\right), \qquad \Phi_n\left(y/b\right) = \sin\left(n\pi y/b\right), \qquad (41)
$$

$$\Lambda_m\left(x/a\right)=\cosh\left(\gamma_m x/a\right)-\cos\left(\gamma_m x/a\right)-\theta_m\left[\sinh\left(\gamma_m x/a\right)-\sin\left(\gamma_m x/a\right)\right],$$
$$\Lambda_n\left(y/b\right)=\cosh\left(\gamma_n y/b\right)-\cos\left(\gamma_n y/b\right)-\theta_n\left[\sinh\left(\gamma_n y/b\right)-\sin\left(\gamma_n y/b\right)\right], \tag{42}$$

where $\theta_m=\left(\cosh\gamma_m-\cos\gamma_m\right)/\left(\sinh\gamma_m-\sin\gamma_m\right)$ and γ_m is the mth root of the frequency equation of a clamped-clamped beam. The assumed trial functions satisfy the following microshell boundary conditions

$$u\big|_{x=(0,a)}=u\big|_{y=(0,b)}=0,\quad v\big|_{x=(0,a)}=v\big|_{y=(0,b)}=0,\quad w\big|_{x=(0,a)}=w\big|_{y=(0,b)}=0,$$
$$\partial w/\partial x\big|_{x=(0,a)}=0,\qquad \partial w/\partial y\big|_{y=(0,b)}=0. \tag{43}$$

Employing Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial \Pi}{\partial q_j}-\frac{\partial W_P}{\partial q_j}=0,\qquad j=1,....,N, \tag{44}$$

yields the discretised nonlinear equations of motion of the spherical microshell. In Eq. (44), q_j stands for the components of the time-dependent generalised coordinates vector ($\mathbf{q}$), given by

$$\mathbf{q}=\left\{u_{m,n},v_{m,n},w_{m,n}\right\}^{\mathrm{T}}. \tag{45}$$

For the *nonlinear static analysis*, a discretised model consisting of 38 symmetric generalised coordinates (due to symmetric configuration of the system and the symmetric external load) is considered (i.e. a 38-degree-of-freedom system). This model consists of 28 in-plane generalised coordinates, namely $u_{2,1}$, $u_{2,3}$, $u_{4,1}$, $u_{2,5}$, $u_{4,3}$, $u_{6,1}$, $u_{2,7}$, $u_{4,5}$, $u_{6,3}$, $u_{8,1}$, $u_{4,7}$, $u_{6,5}$, $u_{8,3}$, and $u_{10,1}$ for u displacement and $v_{1,2}$, $v_{3,2}$, $v_{1,4}$, $v_{5,2}$, $v_{3,4}$, $v_{1,6}$, $v_{7,2}$ $v_{5,4}$, $v_{3,6}$, $v_{1,8}$, $v_{7,4}$, $v_{5,6}$, $v_{3,8}$, and $v_{1,10}$ for v displacement, as well as 10 out-of-plane generalised coordinates , namely $w_{1,1}$,

$w_{1,3}$, $w_{3,1}$, $w_{1,5}$, $w_{5,1}$, $w_{3,3}$, $w_{1,7}$, $w_{7,1}$, $w_{3,5}$ and $w_{5,3}$. For the *linear natural frequency analysis*, a discretised model consisting of 60 generalised coordinates is considered, namely the generalised coordinates (1,1), (2,1), (1,2), (2,2), (3,1), (1,3), (3,2), (2,3), (4,1), (1,4), (3,3), (4,2), (2,4), (5,1), (1,5), (4,3), (3,4), (5,2), (2,5), (4,4) for all *u*, *v*, and *w* motions.

As the case-study for the numerical simulations in this section, the microshell is assumed to be made of Aluminium of E=70 GPa, ν=0.33, and ρ=2700 kg/m^3. The microshell dimensions are a = b = 400 μm and h=2.5 μm. Aluminium characteristic length-scale can be calculated using the data reported in Ref. [8] and the formula suggested in Ref. [9]; hence, for the considered thickness of h=2.5 μm, the length-scale is calculated as 0.9453 μm.

6.1 Linear natural frequency analysis

A linear eigenvalue analysis is performed so as to obtain the effect of the length-scale parameter as well as the radius of spherical microshell on the natural frequencies of the system. In the numerical simulations, the time (t) and natural frequencies ($\hat{\Omega}_{m,n}$) are made dimensionless as

$$t^* = \frac{t}{a^2\sqrt{\rho h/D}}, \quad \Omega_{m,n} = \hat{\Omega}_{m,n}\, a^2\sqrt{\rho h/D}, \tag{46}$$

in which $D = Eh^3/12(1-\nu^2)$.

Figure 3 shows the variation of the natural frequencies ($\Omega_{1,1}$, $\Omega_{2,1}$, and $\Omega_{2,2}$) of the spherical microshell as a function of the radius of curvature. As seen, at large R/a ratios, the rate of change in the natural frequencies decreases; however, at smaller values of the radius of curvature, the natural frequencies vary significantly. Additionally, it is interesting to note that at a certain radius of curvature, the first two natural frequencies (i.e. $\Omega_{1,1}$ and $\Omega_{2,1}$) become the same.

To illustrate the simultaneous effect of the principal radius of curvature and the small-scale effects, Fig. 4 is constructed. Sub-figures (a)-(c) show respectively the variation of $\Omega_{1,1}$, $\Omega_{2,1}$, and $\Omega_{2,2}$ as a function the radius of curvature based on both the modified couple stress and classical continuum theories. As seen, for all the three cases, the difference between the predictions of the two theories becomes less as the radius of curvature decreases. Sub-figure (d) better illustrates this by showing the percentage difference between the two theories in predicting $\Omega_{1,1}$, $\Omega_{2,1}$, and $\Omega_{2,2}$, as a function of the radius of curvature. As seen, the percentage difference between the two theories decreases as the radius of curvature is decreased; in other words, the small-scale effects become weakened as the radius of curvature is decreased. Additionally, it is seen that for the case of $\Omega_{1,1}$, the difference between the two theories reaches a minimum in the vicinity of R/a = 7.

6.2 Nonlinear static analysis

This section analyses the nonlinear static deflection characteristics of the spherical microshell under positive and negative static pressure load. It should be noted that in the numerical simulations, the static load is reported in dimensionless form P, related to its dimensional counterpart through $P = pa^4/(Dh)$.

Figure 5 demonstrates the nonlinear static response of the spherical microshell under static pressure loading in the (a) positive and (b) negative z direction, obtained based on the modified couple stress and classical continuum theories; for this case R/a = 10.0. As seen in sub-figure (a), when the static load is applied in the positive z direction, the difference between the two theories is very small. Additionally, the microshell displays a maximum displacement amplitude of almost $0.88h$ when the static load is applied in the positive z direction. A completely different scenario is observed when the static load is applied in the negative z direction, as shown in sub-figure (b). As seen, the difference between the two theories is significant for this case, however both theories predict two saddle-node bifurcations and an unstable region in between. It is seen that the classical theory predicts a stronger nonlinear behaviour with a larger bistable region. Furthermore, as opposed to the case of sub-figure (a), the microshell displays a maximum displacement of more than $4h$ for the case when the static load is exerted in the negative z direction.

7. Conclusions

The nonlinear formulations for the modified couple stress theory are derived in an orthogonal curvilinear coordinate system. In particular, the expressions for the rotation vector, higher-order strain and stress tensors, i.e. the symmetric rotation gradient tensor and the deviatoric part of the symmetric couple stress tensor, as well as the classical strain and stress tensors are derived in the most general orthogonal curvilinear coordinate system. Utilising the surface theory and the Codazzi-Gauss conditions, the expressions for the MCST are then obtained in a doubly curved orthogonal curvilinear coordinate system, which is specifically suitable for modelling shells of single or double curvature. As special cases, the MCST formulations are also obtained for cylindrical and spherical coordinate systems, which can be readily used for modelling of cylindrical and spherical shells.

As a test case, numerical simulations are conducted for spherical shells and the linear natural frequencies and nonlinear static behaviour are analysed. It is shown that the small-scale effects on the natural frequencies become weakened as the radius of curvature is decreased. Furthermore, it is revealed that the nonlinear static response of the microshell is very sensitive to the direction of the applied load.

References

1. Toupin, R.A.: Elastic materials with couple-stresses. Archive for Rational Mechanics and Analysis **11**, 385-414 (1962)
2. Mindlin, R., Tiersten, H.: Effects of couple-stresses in linear elasticity. Archive for Rational Mechanics and Analysis **11**, 415-448 (1962)
3. Mindlin, R.D.: Micro-structure in linear elasticity. Archive for Rational Mechanics and Analysis **16**, 51-78 (1964)
4. Mindlin, R.D., Eshel, N.N.: On first strain-gradient theories in linear elasticity. International Journal of Solids and Structures **4**, 109-124 (1968)
5. Yang, F., Chong, A.C.M., Lam, D.C.C., Tong, P.: Couple stress based strain gradient theory for elasticity. International Journal of Solids and Structures **39**, 2731-2743 (2002)
6. Park, S.K., Gao, X.L.: Bernoulli–Euler beam model based on a modified couple stress theory. Journal of Micromechanics and Microengineering **16**, 2355 (2006)
7. Eringen, A.C.: Mechanics of continua: Wiley (1967)
8. Haque, M.A., Saif, M.T.A.: Strain gradient effect in nanoscale thin films. Acta Materialia **51**, 3053-3061 (2003)
9. Chen, S.H., Feng, B.: Size effect in micro-scale cantilever beam bending. Acta Mechanica **219**, 291-307 (2011)

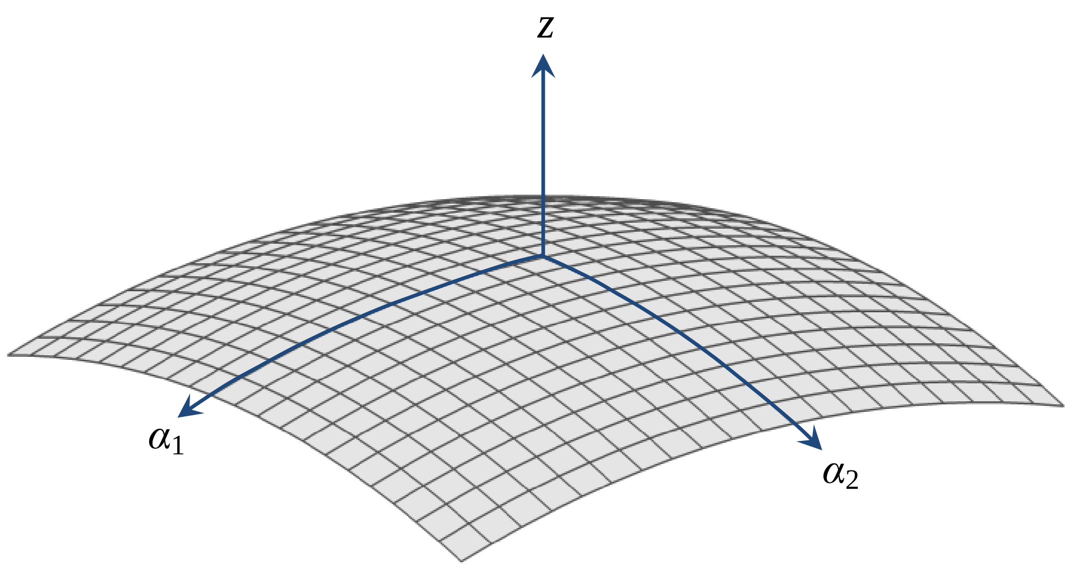


Figure 1. Schematic representation of an orthogonal doubly curved coordinate system.

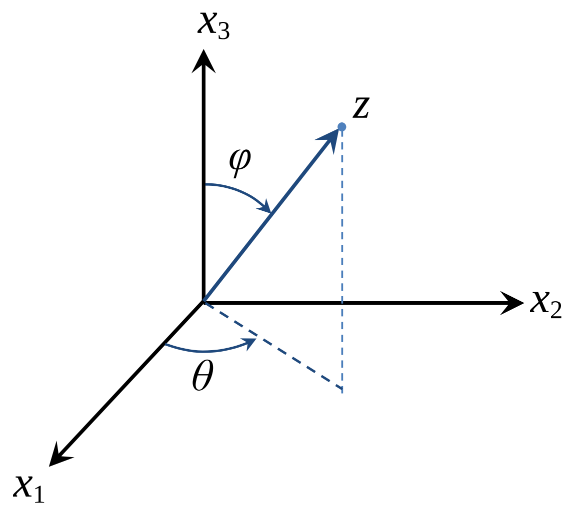


Figure 2. Schematic representation of a spherical coordinate system.

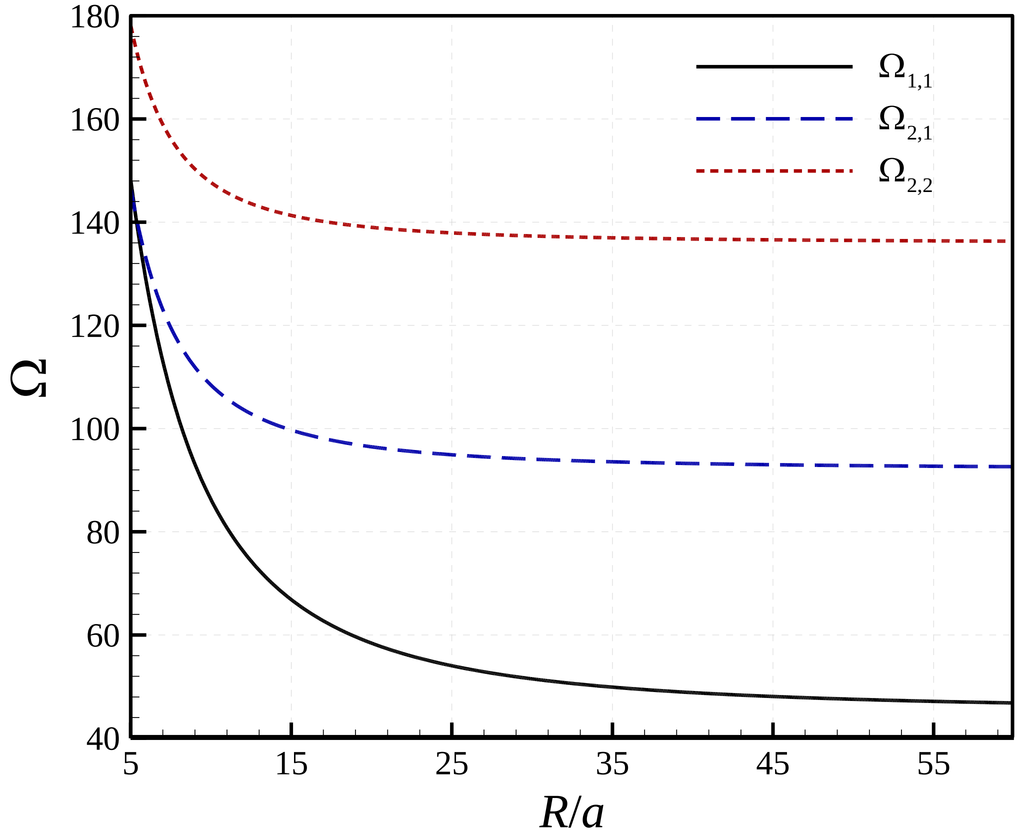
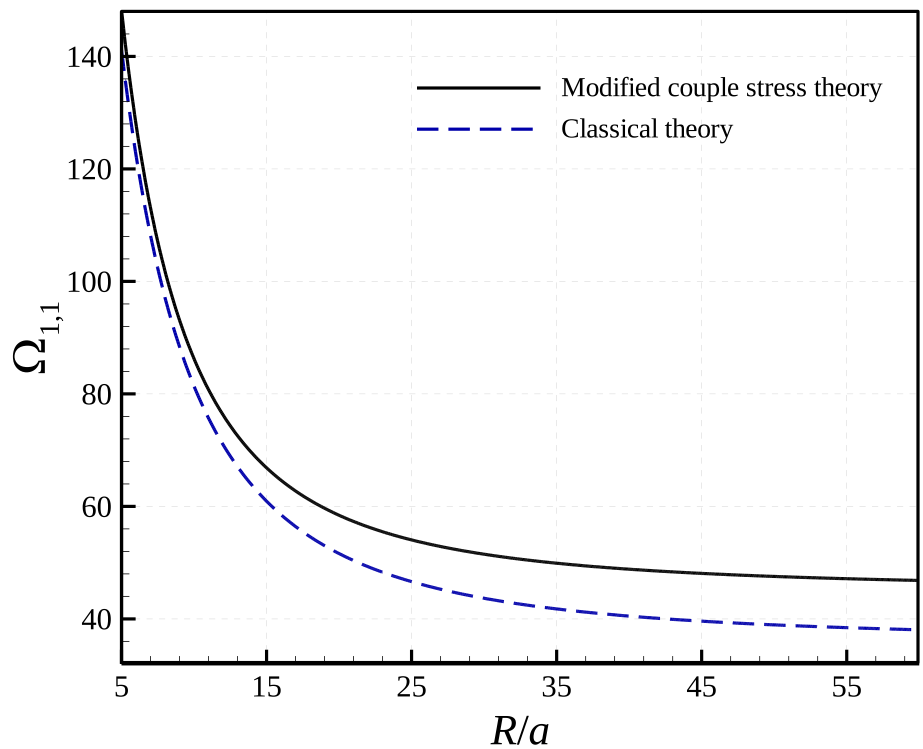


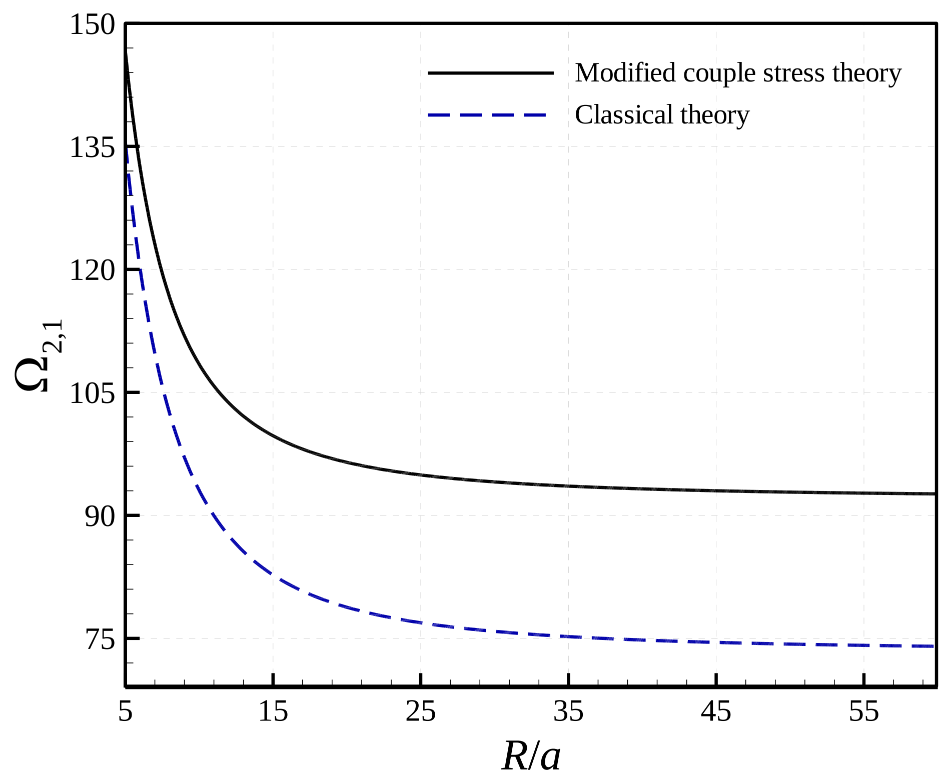
Figure 3: The first three dimensionless out-of-plane natural frequencies of the spherical microshell predicted by the modified couple stress theory as a function of the radius of curvature.

(a)



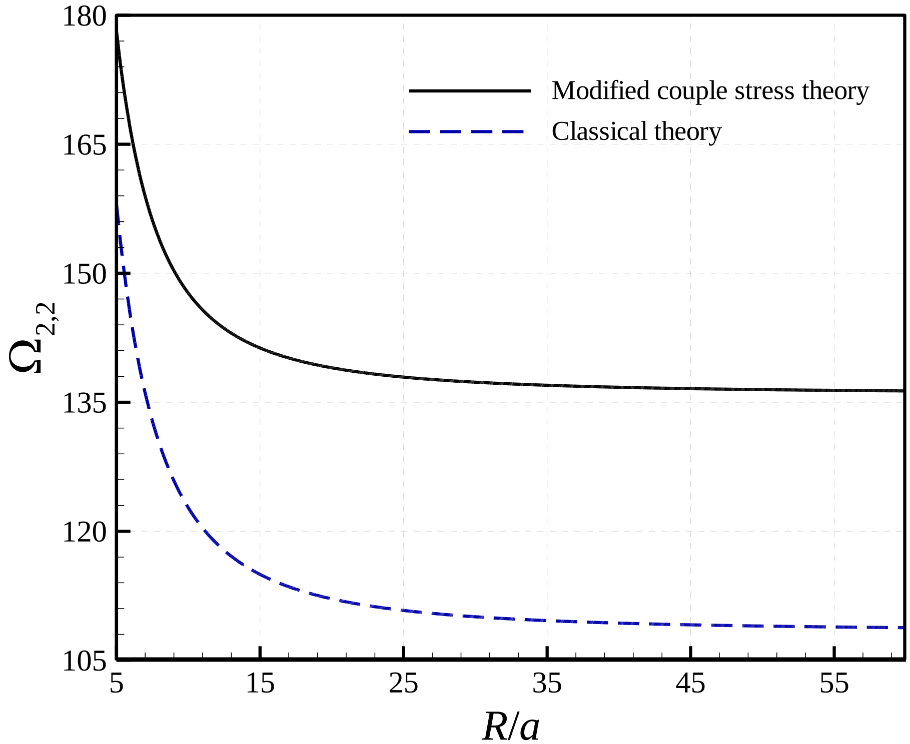
Modified couple stress theory
Classical theory
140
120
100
80
60
40
$\Omega_{1,1}$
5
15
25
35
45
55
R/a

(b)



Modified couple stress theory
Classical theory
150
135
120
105
90
75
$\Omega_{2,1}$
5
15
25
35
45
55
R/a

(c)



(d)

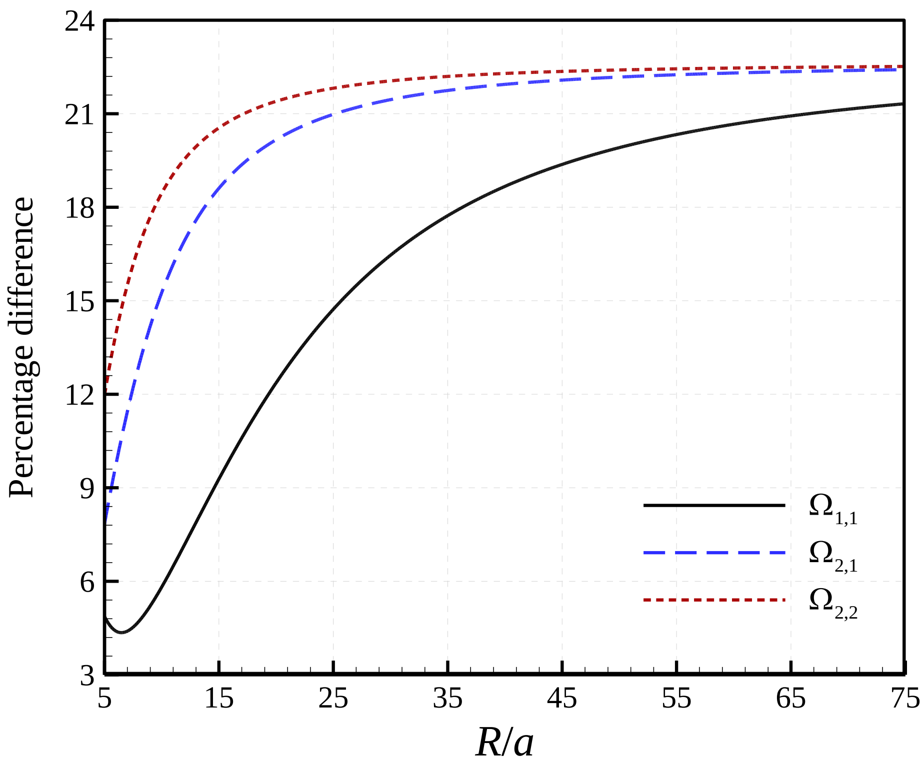
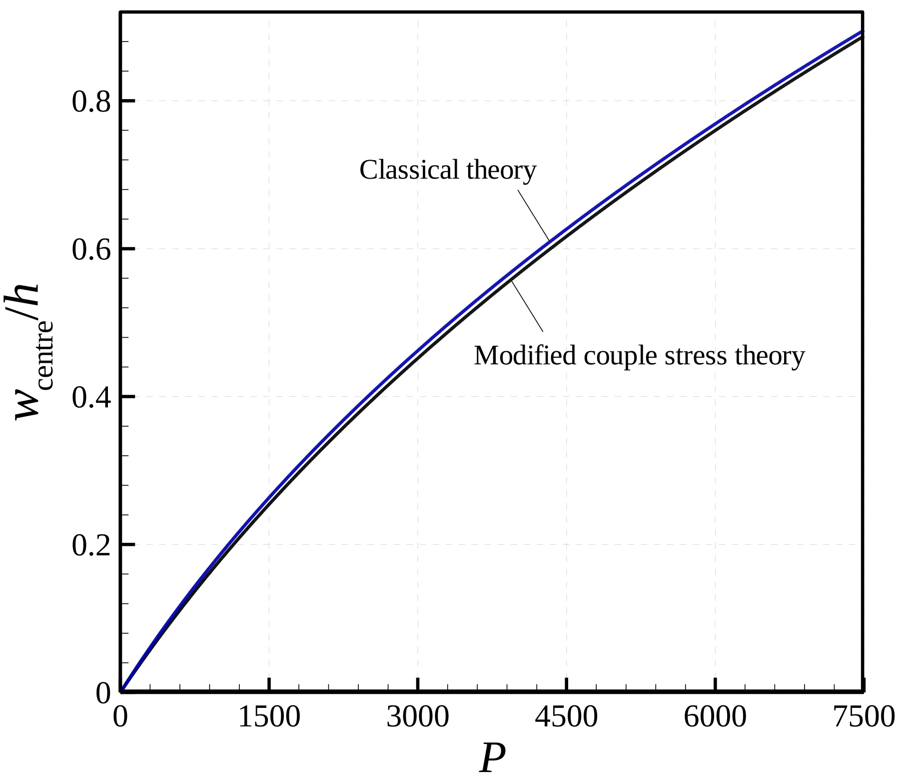


Figure 4: The dimensionless out-of-plane natural frequencies of the spherical microshell predicted by the modified couple stress and classical theories as a function of the radius of curvature: (a) $\Omega_{1,1}$, (b) $\Omega_{2,1}$, and (c) $\Omega_{2,2}$. (d) the percentage difference between the natural frequencies predicted by the two theories, as a function of the radius of curvature.

(a)



(b)

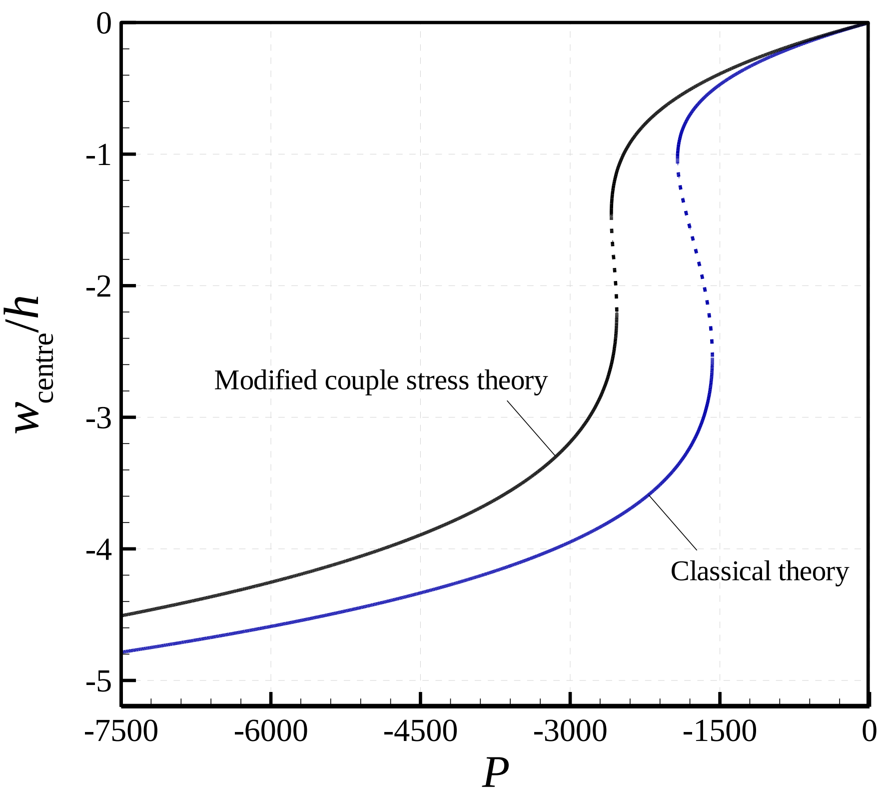


Figure 5: Comparison between the nonlinear static response of the microshell, under (a) positive and (b) negative pressure load, predicted by the modified couple stress and classical theories. R/a = 10.0.